

Phase transitions in full counting statistics for periodic pumping

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(Dated: July 14, 2010)

We discuss the problem of full counting statistics for periodic pumping. The probability generating function is usually defined on a circle of the “physical” values of the counting parameter, with its periodicity corresponding to charge quantization. The extensive part of the generating function can either be an analytic function on this circle or have singularities. These two cases may be interpreted as different thermodynamic phases in time domain. We discuss several examples of phase transitions between these phases for classical and quantum systems. Finally, we prove a criterion for the “analytic” phase in the problem of a quantum pump for noninteracting fermions.

I. INTRODUCTION

The problem of full counting statistics (FCS)^{1,2} is often considered in a setup periodic in time.³ In such a formulation, the system depends on external parameters varying periodically in time, and one is interested in counting certain quantized events (typically, the transfer of particles between the leads of a contact). Generally, the probabilities of different outcomes P_n are labeled by integer indices n and can be combined into the probability generating function^{1,4}

$$\chi(\lambda) = \sum_{n=-\infty}^{+\infty} P_n e^{i\lambda n}. \quad (1)$$

For a superposition of statistically independent processes, the generating function is given by the product of those for each process. Therefore, for a periodic process extended in time, this generating function is exponentially extensive in time, provided the correlations in time decay sufficiently rapidly. One therefore usually defines the generating function “per period”³ (or “extensive” FCS)

$$\chi_0(\lambda) = \exp \left[\lim_{N \rightarrow \infty} \frac{1}{N} \ln \chi(\lambda) \right], \quad (2)$$

where the FCS $\chi(\lambda)$ is collected over N periods.

A vast literature is devoted to FCS both in the general setup and for periodic processes.² Probably, the most interesting class of FCS problems are those of quantum charge transfer,^{1,3} but there are also discussions of FCS in classical stochastic processes (see, e.g., Ref. 5) and of the relation between classical and quantum effects in FCS (see, e.g., Sec. 5 of Ref. 4 and Ref. 6).

From this immense body of results, one notices that FCS in time-extensive problems can be conveniently classified by the analytic properties of the FCS generating function per period (2). Namely, two main phases can be identified: $\ln \chi_0(\lambda)$ may either be analytic on the unit circle of real λ or have a singularity at certain values of λ . Correspondingly we distinguish two phases of FCS: *analytic* and *nonanalytic*.

In the present paper we show that the existence of these two phases is a very general feature of FCS: phase transitions between them occur both in classical stochastic models and in quantum systems. We illustrate our discussion with several examples: the classical weather model and quantum systems of noninteracting fermions. For quantum examples, we use the results of our recent work^{7,8} to show the stability of the analytic phase for a wide class of non-interacting fermionic systems, even at finite temperature. Finally, we discuss a possible identification of the two phases in terms of cumulants and conjecture the general form of the asymptotic long-time behavior of the FCS in the nonanalytic phase.

II. TWO PHASES OF FCS IN TIME-PERIODIC SYSTEMS

The generating function (1) is periodic, $\chi(\lambda) = \chi(\lambda + 2\pi)$, and its Fourier components P_n are non-negative. Furthermore, the probabilities P_n always obey the normalization condition $\sum_n P_n = 1$, and therefore for non-negative P_n the series (1) converges uniformly on the unit circle $|e^{i\lambda}| = 1$. Typically, P_n decay sufficiently rapidly as functions of n , and then $\chi(\lambda)$ is analytic for real values of λ .

Out of the above three properties (periodicity, non-negativity of probabilities and analyticity) only the first one (periodicity) necessarily holds for the generating function per period (2) from its definition. As we shall see below, both the non-negativity of the Fourier components (“quasiprobabilities”) and the analyticity may be broken for $\chi_0(\lambda)$. Note that non-negativity of the quasiprobabilities is not always related to analyticity (e.g., $\chi_0(\lambda)$ may be analytic and still have some negative quasiprobabilities).

As we shall see below, most common nonanalytic features of $\chi_0(\lambda)$ are discontinuities and kink points. In those cases, the quasiprobabilities decay algebraically in n and oscillate in sign (if the singularity is located at $\lambda \neq 0$). On the other hand, an analytic $\chi_0(\lambda)$ corresponds to an exponential decay of the quasiprobabilities.

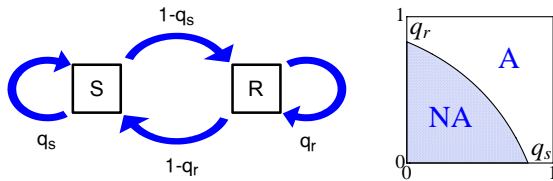


FIG. 1: (a) Weather model. Transitions to sunny (rainy) weather S (R) are shown by arrows with corresponding probabilities. (b) Phase diagram of the weather model in the (q_s, q_r) parameter space. The shaded area corresponds to the nonanalytic phase (NA).

ties (not necessarily non-negative). Thus we may view the difference between the analytic and nonanalytic behaviors of $\chi_0(\lambda)$ as a distinction between two quantum phases. As usual for phase transitions, the transition to the nonanalytic phase appears only in the thermodynamic limit (when one considers the extensive part of the generating function).

We may also remark in passing that there is nothing surprising in negative quasiprobabilities in the extensive generating function $\chi_0(\lambda)$. Indeed, this generating function appears in our attempt to factorize the full FCS into N independent processes (2). Generally, there is no reason to expect that such decomposition is possible with physical non-negative probabilities, since the system has certain correlations between different periods.

III. CLASSICAL EXAMPLE: WEATHER MODEL

The simplest stochastic process that illustrates the analytic-nonanalytic phase transition is the so called “weather model”.⁹ Suppose that the weather on each day can be of two types: rainy or sunny, and that the weather on a given day is chosen randomly with the probabilities depending on the weather on the previous day. Let the probability of a sunny day after a sunny day be q_s and the probability of a rainy day followed by a rainy day be q_r (Fig. 1a). This defines a stochastic process, in which the FCS for the number of sunny days can be easily calculated.¹⁰ The calculation involves two generating functions $\chi_s(\lambda)$ and $\chi_r(\lambda)$, which correspond to the conditions that the last day is sunny or rainy, respectively. The evolution of these generating functions is described by a linear operator acting in the two-dimensional space. The full generating function is then given by

$$\chi(\lambda) = \chi_s(\lambda) + \chi_r(\lambda) = C_1[\omega_1(\lambda)]^N + C_2[\omega_2(\lambda)]^N, \quad (3)$$

where N is the number of observation days, $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are the two eigenvalues of the evolution operator, and C_1 and C_2 are some coefficients. Therefore

$$\chi_0(\lambda) = \max(\omega_1(\lambda), \omega_2(\lambda)), \quad (4)$$

where the eigenvalue with the maximal absolute value is chosen. Then two phases are possible: either one eigen-

value remains leading for all values of λ (analytic phase) or the leading eigenvalue switches at some value of λ (nonanalytic phase). From the explicit formula for the eigenvalues

$$\omega_{1,2} = \frac{q_r + q_s e^{i\lambda}}{2} \pm \sqrt{\left(\frac{q_r + q_s e^{i\lambda}}{2}\right)^2 + (1 - q_s - q_r)e^{i\lambda}} \quad (5)$$

one can deduce the phase diagram in the (q_s, q_r) coordinates, (Fig. 1b) and find that the nonanalyticity occurs at $\lambda = \pi$.

IV. QUANTUM EXAMPLES: NON-INTERACTING FERMIONS

As quantum examples illustrating the analytic-nonanalytic transition in time-periodic FCS, we consider systems of non-interacting fermions. This class of systems has been studied extensively,² with the main result given by the so-called Levitov–Lesovik determinant formula for the characteristic function (1).^{1,3,11,12} In the periodic case, the same determinant formula can be used to find the extensive part of the FCS (2) by simply imposing periodic boundary conditions in time.³

For simplicity, we consider here a one-channel quantum contact of non-interacting spinless fermions. The transparency of the contact g is taken to be time-independent, but we assume a time-dependent voltage $V(t)$ applied to the contact and an arbitrary temperature T . This system was considered in previous works,¹³ and we can use their results to study the analytic-nonanalytic transition in a periodic setup. The case of a more general time-dependent scattering matrix^{7,8} will be treated in the next section.

The FCS for non-interacting fermions can be expressed in terms of the distribution function of effective transparencies $\mu(p)$ for single-particle processes.⁸ This function determines the jump of the derivative of $\ln \chi_0(\lambda)$ on the negative real axis of the variable $e^{i\lambda}$ [so that $p \in (0, 1)$]:

$$\mu(p) = \frac{1}{2\pi i} \partial_p \ln \chi(\lambda) \Big|_{p-i0}^{p+i0}, \quad p = \frac{1}{1 - e^{i\lambda}}. \quad (6)$$

Since, in the non-interacting case, singularities of $\chi_0(\lambda)$ are allowed only on the negative real axis of $e^{i\lambda}$ (i.e., if we take λ to be real, at $\lambda = \pi$),^{7,8} the analytic and nonanalytic phases correspond to $\mu(p = 1/2) = 0$ and $\mu(p = 1/2) \neq 0$, respectively.

From the previous studies of the one-channel contact with a time-independent transparency g , we can sketch the general behavior of the distribution function $\mu(p)$ in several special cases (Fig. 2):

(a) $T = 0$, $V(t) = \text{const}$. In this case $\mu(p) = \alpha \delta(p - g)$ or $\mu(p) = \alpha \delta(p - [1 - g])$, depending on the polarity of V (Fig. 2a). The weight $\alpha = V\tau/(2\pi)$, where τ is the conventional period. This case obviously belongs to the *analytic* phase, unless $g = 1/2$.

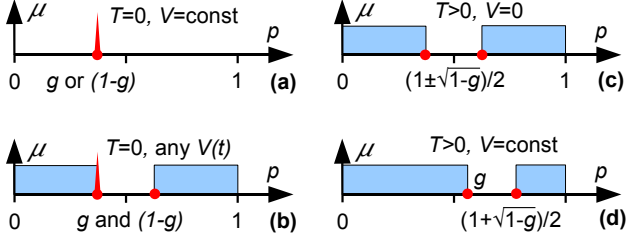


FIG. 2: Schemes of distributions of effective transparencies $\mu(p)$ for different examples of FCS in a quantum contact. Shaded areas represent a continuous spectrum $\mu(p)$. Peaks at gap edges schematically represent delta-function contributions.

(b) $T = 0$, arbitrary $V(t)$. In this case, $\mu(p)$ is given by a superposition of the delta function from the previous example and a generally continuous spectrum for $|p - 1/2| \geq |g - 1/2|$ (gapped around $p = 1/2$, except for $g = 1/2$), see Fig. 2b. This form of the spectrum (delta function and a continuous part) follows from the arguments in Refs. 7,13). In addition, it is proved there that the continuous part is symmetric with respect to $p = 1/2$. The existence of the gap around $p = 1/2$ also follows from the theorem proved in the next section. Except for $g = 1/2$, this case also belongs to the *analytic* phase.

(c) $T > 0$, $V(t) = 0$. In this case, the spectrum $\mu(p)$ is symmetric, continuous, and gapped around $p = 1/2$ (Fig. 2c). The value of the gap can be obtained by a direct calculation^{12,14}: $|p - 1/2| \geq \sqrt{1-g}/2$ (also predicted by the theorem proven in the next section). This example belongs to the *analytic* phase as well.

(d) $T > 0$, $V(t) = \text{const}$ (without loss of generality, we assume $V > 0$, otherwise we may simply reflect $p \mapsto 1-p$). In this case, a calculation shows that the spectrum is continuous and consists of two regions: $0 \leq p \leq g$ and $(1 + \sqrt{1-g})/2 \leq p \leq 1$ (Fig. 2d). This case belongs to the *analytic* phase, if $g < 1/2$ and to the *nonanalytic* phase, if $g > 1/2$. Note that the two regions of support of $\mu(p)$ overlap if $g \geq 3/4$.

(e) $T > 0$, arbitrary $V(t)$. In this case, the spectrum $\mu(p)$ is generally continuous and non-symmetric. As we prove in the theorem in the next section, the *analytic* phase is realized for $g < 1/2$ (which results in the gap $|p - 1/2| > 1/2 - g$). The *nonanalytic* phase is generic for $g > 1/2$.

V. ANALYTIC PHASE FOR NON-INTERACTING QUANTUM SYSTEMS

Let us now consider a slightly more general setup: a one-channel two-lead contact which can be described by an instantaneous time-dependent 2×2 scattering matrix $S(t)$. We assume a finite temperature T . The character-

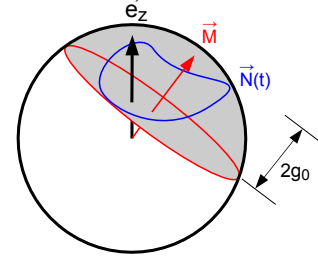


FIG. 3: Spherical cap of height $2g_0$ covering the trajectory $\vec{N}(t)$ and the north pole \vec{e}_z .

istic function $\chi(\lambda)$ in this case is given by⁸

$$\chi(\lambda) = \det \left(\left[1 + (e^{i\lambda} - 1)\tilde{X} \right] e^{-i\lambda Q} \right). \quad (7)$$

Here the determinant is taken in the single-particle Hilbert space, $Q = (1 + \sigma^z)/2$ is the charge operator in one of the leads, and the Hermitian operator \tilde{X} is defined as (n_F is the Fermi occupation number)

$$\tilde{X} = (1 - n_F)Q + (n_F)^{1/2} S^\dagger Q S (n_F)^{1/2}. \quad (8)$$

The spectrum of the operator \tilde{X} is real and confined to the interval $[0, 1]$. Its eigenvalues can be interpreted as effective transparencies of the contact between leads, and its spectral density $\mu(p) = \text{tr} \delta(p - \tilde{X})$ determines the characteristic function as

$$\ln \chi(\lambda) = -i\lambda \text{Tr} Q + \int_0^1 dp \mu(p) \ln [1 + (e^{i\lambda} - 1)p]. \quad (9)$$

The same calculation applies to the extensive part of the generating function $\chi_0(\lambda)$, if one imposes periodic boundary conditions in time.³

As mentioned in the previous section, the analytic or nonanalytic phase in the periodic setup depends on whether $\mu(p = 1/2)$ is zero or non-zero. The proof of the analytic phase thus amounts to demonstrating a gapped region around $p = 1/2$ for a certain class of time-dependencies $S(t)$. Namely, we prove below that if the trajectory of $S(t)$ is confined within a certain region (spherical cap), then there is a gap in the spectrum $\mu(p)$ around $p = 1/2$. Consider the matrix $\hat{N} = S^\dagger \sigma^z S$ or, equivalently, the vector $\vec{N} = (1/2) \text{Tr} \vec{\sigma} \hat{N}$ (it is a time-dependent vector on the unit sphere¹⁵). We suppose that this vector, at all times, together with the north pole $\vec{N} = \vec{e}_z$, fit in a certain spherical cap of height $2g_0 < 1$ (see Fig. 3). Then we can prove that $\mu(p) = 0$ within the window $|p - 1/2| < 1/2 - g_0$.

Indeed, the above assumption may be expressed mathematically as the existence of a constant unit vector \vec{M} such that

$$\vec{M} \vec{N}(t) \geq 1 - 2g_0 \quad \text{for all } t \quad \text{and} \quad \vec{M} \vec{e}_z \geq 1 - 2g_0. \quad (10)$$

We further denote $\hat{M} = \vec{M} \vec{\sigma}$ and assume that n_F is a scalar in the lead space (the same temperature in both

leads), in particular $[n_F, \hat{M}] = [n_F, \sigma^z] = 0$. Then, after a simple algebra,

$$\begin{aligned} (2\tilde{X} - 1)^2 &\geq -(\alpha\hat{M})^2 + \{2\tilde{X} - 1, \alpha\hat{M}\} \\ &= -\alpha^2 + \alpha \left(\{\hat{M}, \sigma^z\}(1 - n_F) + n_F^{1/2} \{\hat{M}, \hat{N}\} n_F^{1/2} \right). \end{aligned} \quad (11)$$

This inequality holds for any real α . In particular, we can choose $\alpha = 1 - 2g_0 \geq 0$. If we now use the fact that the eigenvalues of n_F are all located between 0 and 1, we arrive at the desired inequality

$$(\tilde{X} - 1/2)^2 \geq (1/2 - g_0)^2. \quad (12)$$

This result implies a gap in $\mu(p)$ of the size $1 - 2g_0$ centered around $p = 1/2$, i.e., the *analytic* phase.

The class of trajectories for which our theorem applies is quite wide. One particular case are those trajectories, for which the *instantaneous* transparencies $g(t) = (1 - \vec{N}(t)\vec{e}_z)/2$ are bounded from above by a certain maximal transparency $g_0 < 1/2$. In this case, one may safely choose $\hat{M} = \sigma^z$ and prove the analytic phase.

Interestingly, in the example (c) of the previous section ($T > 0$ and time-independent S), it is more advantageous to tilt the vector \vec{M} so that it bisects the angle between \vec{e}_z and \vec{N} : this explains the gap $\sqrt{1 - g}$ in the spectrum $\mu(p)$ in this example.^{12,14}

Another interesting particular case is $T = 0$. In this case, $\mu(p)$ is invariant with respect to global rotations of the trajectory $\vec{N}(t)$ (see Ref. 15). This implies that the condition $\vec{M}\vec{e}_z \geq 1 - 2g_0$ does not need to be taken into account: it suffices to cover only the trajectory $\vec{N}(t)$ (and not the north pole) by any spherical cap smaller than hemisphere, in order to prove the analytic phase. In particular, the transparencies g and $1 - g$ produce the same gaps in examples 1 and 2 of the previous section.

Note that the above theorem may also be extended to the case of different temperatures of both leads. In this case, we are restricted to $\hat{M} = \sigma^z$ (the condition $[n_F, \sigma^z] = 0$ is still assumed as the absence of initial entanglement of the leads⁸).

Finally, the theorem also applies to the multichannel problem, with the only modification that the conditions (10) should be replaced by $(1/2)\{\hat{M}, \hat{N}(t)\} \geq 1 - 2g_0$ and $(1/2)\{\hat{M}, \sigma^z\} \geq 1 - 2g_0$, in terms of all eigenvalues.

VI. DISCUSSION OF RESULTS

We have identified two phases of FCS, analytic and nonanalytic, in terms of the analyticity of the generating function per period (2) at real values of the counting parameter λ . Although singularities of $\chi_0(\lambda)$ in all examples considered in the present paper appeared at $\lambda = \pi$, they may generally occur at any values of λ . We suggest that the appearance of those singularities may be

the most common scenario of the analytic-nonanalytic phase transition and ask the natural question: what are the fingerprints of such a transition?

The answer to this question may be deduced from analyzing classical models (e.g., the weather model described above). Let us define the *staggered average*: $\langle A(n) \rangle_\pi = \langle (-1)^n A(n) \rangle / \langle (-1)^n \rangle$ for any function of the counted events $A(n)$.²⁰ Then the *staggered cumulant*

$$\langle \langle n^2 \rangle \rangle_\pi = \langle n^2 \rangle_\pi - \langle n \rangle_\pi^2 = (-i\partial_\lambda)^2 \ln \chi(\lambda)|_{\lambda=\pi} \quad (13)$$

has different asymptotics as a function of the number of periods N in the two phases. In the *analytic* phase, $\langle \langle n^2 \rangle \rangle_\pi \propto N$, while in the *nonanalytic* phase, two eigenvalues $\omega_{1,2}(\lambda)$ contribute, and one finds $\langle \langle n^2 \rangle \rangle_\pi \propto N^2$.

In quantum noninteracting systems, the same distinction between the two phases can be made if the asymptotic behavior of the generating function $\chi(\lambda)$ over N periods is given in the *nonanalytic* phase by

$$\chi(\lambda) = C_1[\chi_0(\lambda - 0)]^N + C_2[\chi_0(\lambda + 0)]^N, \quad (14)$$

where $\lambda \pm 0$ refers to taking the analytic branches below and above the cut at $\lambda = \pi$. The coefficients $C_{1,2}$ are non-extensive (sub-exponential in N) and generally depend on λ . In some situations (e.g., for transmission of non-interacting fermions through an open channel or for counting one-dimensional fermions on a line segment at zero temperature), these coefficients are known to have an algebraic time dependence, $C_{1,2} \propto N^{\gamma(\lambda \mp 0)}$, and thus contribute logarithmically to cumulants.¹⁶ In fact, in these simple examples, the expression (14) can be rigorously proven using the extended Fisher-Hartwig conjecture (theorem),^{17,18} as the FCS in those cases is given exactly by a Toeplitz determinant.

We may further conjecture that the expression (14), with $C_{1,2}$ algebraically depending on N , should remain valid for a larger class of models: for example, for all non-interacting fermionic systems,⁸ where the FCS problem can be reduced to the determinant of a block-Toeplitz matrix. It may even be possible that this result is applicable for interacting systems. These interesting problems remain for future study.

Note that a phase transition in FCS in a particular interacting system has been recently discussed in Ref. 19, with the difference that their phase transition reveals itself in the ordinary (non-staggered) cumulants and is likely connected to a nonanalyticity in $\chi_0(\lambda)$ at $\lambda = 0$ (cf. $\lambda = \pi$ in our examples). One can also invent other examples of phase transitions with singularities appearing at various values of λ . For example, in a superconducting system with the charge transfer quantized in pairs of electrons, a singularity would appear at $\lambda = \pi/2$, (see example V.A in Ref. 8).

VII. ACKNOWLEDGMENTS

We have benefited from discussions with T. Giamarchi, L. Levitov, A. Mirlin, and E. Sukhorukov. A.G.A. is

grateful to ITP, EPFL for hospitality. A.G.A. was sup-

ported by the NSF under the grant DMR-0906866.

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